Saber, Universidad de Oriente, Venezuela.Vol. 18. Nº 2: 207-211(2006)

A LEVINSON TYPE ALGORITHM FOR VANDERMONDE SYSTVEMS

UN ALGORITMO TIPO LEVINSON PARA SISTEMAS DE VANDERMONDE

Wadie Aziz; Teodoro Lara

Universidad de los Andes, Núcleo Universitario "Rafael Rangel",
Departamento de Física y Matemática, Trujillo - Venezuela.
wadie@ula.ve, tlara@ula.ve.

ABSTRACT

In this paper work we examine a Levinson type technique for solving dual Vandermonde systems by means of block matrix and block vector operations which reduces the number of operations, save memory space considerably, in this sense, the algorithm introduced here improve previous ones.

Key word: Vandermode matrix, Vandermode systems, Levinson.

RESUMEN

En este trabajo examinamos una técnica tipo Levinson para resolver los sistemas de Vandermonde por medio de operaciones matriciales en bloques y vectoriales en bloques los cuales reduce el número de operaciones, optimizando considerablemente el espacio de memoria, en este sentido, este algoritmo mejora el existente.

Palabras clave: Matriz Vandermonde, sistemas Vandermonde, Levinson.

INTRODUCTION

Several applications like interpolation and approximation problems drive into linear systems of equations where the matrix involved (or its transpose) is a Vandermonde matrix; that is, the system to be solved

(1) $\mathbf{V}\, x = b$

or its dual

(2) $\mathbf{V}^T \mathrm{a} = \mathrm{f}$

Involves the matrix **V** which is Vandermonde matrix of order n defined by n different scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$;

$$V(\alpha_1, \alpha_2, ..., \alpha_n) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{pmatrix},$$

We shall consider dual Vandermonde systems (2), where the solution is obtained from the Newton formula for polynomial interpolation of points α_i, f_i. We will perform a matrix reformulation of this algorithm where is possible to calculate a decomposition **UL** of $\mathbf{V}^{-T}$ by developing a recurrent algorithm for the dual system; this will be performed by using Levinson type techniques (Porsani, 1992). These techniques are based on the knowledge of the solution of the dual Vandermonde system of order k and the one corresponding to a particular system of order k . Making a suitable block decomposition of the Vandermonde matrix of order (k+1). We find the solution of the (k+1)-system.

In Porsani, an algorithm, using Levinson type techniques, was derived in order to solve the dual Vandermonde system

(3) $V_n^T \mathrm{a} = \mathrm{f}$,

Where $\mathbf{V}_n$ is the n^{th}-order Vandermonde matrix

$$V_n = V(\alpha_1, \alpha_2, \dots, \alpha_n).$$

The description we make of this algorithm uses a different approach from the one given in Gene *et al.* We take into account, as fundamental operations, block vector and block matrix operations. In order to do that, the solution of a particular system

(4) $V_n^T \ell = -\vartheta$,

With $\vartheta = (\alpha_1^n, \alpha_2^n, \cdots, \alpha_n^n)^T$, is needed.

Recibido: abril 2005. Aprobado: enero 2006.
Versión final: febrero 2006

These types of systems arise in many applications; among others, we can mention polynomial fitting, signal processing and encoding theory. We notice that $n\times n$– Vandermonde matrix is completely determined by n arbitrary elements $\alpha_1,\alpha_2,\cdots,\alpha_n$ We take a short look at the algorithm given in (Porsani 1992). We reformulate it in terms of block matrices operations in such a way that null elements are allowed in the Vandermonde matrix. Also, we show that this algorithm works for any set of α_i's.

SOLVING A PARTICULAR VANDERMONDE SYSTEM

As it was said before, system (4) will be solved by means of a recurrent formula which relates the solution of a subsystem of order k, that is,

$$(5)\qquad V_k^T \ell^{(k)} = -\vartheta^{(k)}$$

Where $\ell^{(k)} = (\ell_1^{(k)},\ell_2^{(k)},\cdots,\ell_k^{(k)})^T$ is its solution, with the solution of a subsystem of order (k+1).

The algorithm consists in assuming that the solution $\ell^{(k)}$ of (5) is know and computing $\ell^{(k+1)}$ by solving equation

$$(6)\qquad V_{k+1}^T \ell^{(k+1)} = -\vartheta^{(k+1)}.$$

In matrix form, we have

$$\begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{k-1} & \alpha_1^k \\ \vdots & & & & \\ 1 & \alpha_k & \cdots & \alpha_k^{k-1} & \alpha_k^k \\ \hline 1 & \alpha_{k+1} & \cdots & \alpha_{k+1}^{k-1} & \alpha_{k+1}^k \end{pmatrix} \begin{pmatrix} \ell_1^{(k+1)} \\ \vdots \\ \ell_k^{(k+1)} \\ \ell_{k+1}^{(k+1)} \end{pmatrix} = -\begin{pmatrix} \alpha_1^{k+1} \\ \vdots \\ \alpha_k^{k+1} \\ \alpha_{k+1}^{k+1} \end{pmatrix},$$

or, more which can be expressed in a compact form as follows

$$\begin{pmatrix} 1_k & R_k \\ 1 & u_k^t \end{pmatrix} \begin{pmatrix} \ell_1^{(k+1)} \\ h_k \end{pmatrix} = -\begin{pmatrix} g_k \\ \alpha_{k+1}^{k+1} \end{pmatrix},$$

Where the components of $\mathbf{h}_k$ are the last k^{th} components of $\ell^{(k+1)}$, $g_k = (\alpha_1^{k+1}, \alpha_2^{k+1}, \ldots, \alpha_k^{k+1})^t$, $u_k^t = (\alpha_{k+1}, \alpha_{k+1}^2, \ldots, \alpha_{k+1}^k)$, and $\mathbf{R}_k$ is the upper right block matrix.

By performing the block multiplication we came out with

$$(7)\qquad \begin{cases} \ell_1^{(k+1)}.\ 1_k + R_k.\ h_k = -g_k \\ \ell_1^{(k+1)} + u_k^t.\ h_k = -\alpha_{k+1}^{k+1}. \end{cases}$$

We get $\mathbf{h}_k$ from the first relation since $\mathbf{R}_k$ is an invertible matrix

$$\begin{aligned} h_k &= R_k^{-1}\ (-g_k - \ell_1^{(k+1)}.\ 1_k) \\ &= R_k^{-1}\ (-g_k) + \ell_1^{(k+1)}\ R_k^{-1}\ (-1_k). \end{aligned}$$

We obtain $\mathbf{R}_k\cdot x = -g_k$ multiplying (5) by α_i for arbitrary i In consequence both systems have the same solution.

Therefore

$$(8)\qquad h_k = \ell^{(k)} + \ell_1^{(k+1)} t^{(k)}, \text{where } t^{(k)} = R_k^{-1}(-1_k).$$

Later on we will solve system

$$(9)\qquad R_k.t^{(k)} = -1_k$$

and will see that its solution is completely determined by $\ell^{(k)}$.

In order to find $\ell_1^{(k+1)}$, we plug $\mathbf{h}_k$ into the second equation in (7) and then solve for $\ell_1^{(k+1)}$,

$$(10)\qquad \ell_1^{(k+1)} = -\frac{\alpha_{k+1}^{k+1} + u_k^t.\ell^{(k)}}{1 + u_k^t.t^{(k)}}.$$

In conclusion, from (8) and (10), the solution of (6) becomes

$$(11)\qquad \ell^{(k+1)} = \begin{pmatrix} 0 \\ \ell^{(k)} \end{pmatrix} + \ell_1^{(k+1)} \begin{pmatrix} 1 \\ t^{(k)} \end{pmatrix}.$$

Lemma 2.1 $1 + u_k^t.t^{(k)} \neq 0$

Proof: For $\alpha_{k+1}=0$ result is true. If $\alpha_{k+1} \neq 0$ and $t^{(k)}$ verifies

$$1 + u_k^t.t^{(k)} = 0$$

then the k – degree polynomial

$$p(x) = 1 + t_1^{(k)} x + \cdots + t_k^{(k)} x^k$$

would have α_i,i =,..., k as simple roots. So $P(\alpha_{k+1}) \neq 0$ and therefore,

$$1 + u_k^t.t^{(k)} \neq 0.$$

On the other hand, the system

(12) $\mathbf{R}_{k+1} \cdot \mathbf{t}^{(k+1)} = -1_{k+1}$,

written down in block form looks like

$$\begin{pmatrix} \mathrm{R}_k & \mathrm{g}_k \\ \mathrm{u}_k^t & \alpha_{k+1}^{k+1} \end{pmatrix} \begin{pmatrix} \mathrm{s}_k \\ t_{k+1}^{(k+1)} \end{pmatrix} = -\begin{pmatrix} 1_k \\ 1 \end{pmatrix},$$

where $\mathrm{g}_k = (\alpha_1^{k+1}, \dots, \alpha_k^{k+1})^t$, $\mathrm{s}_k = (t_1^{(k+1)}, \dots, t_k^{(k+1)})^t$ and this is equivalent to

$$(13) \quad \begin{cases} \mathrm{R}_k \cdot \mathrm{s}_k + \mathrm{g}_k \cdot t_{k+1}^{(k+1)} = -1_k \\ \mathrm{u}_k^t \cdot \mathrm{s}_k + \alpha_{k+1}^{k+1} \cdot t_{k+1}^{(k+1)} = -1. \end{cases}$$

As $\mathbf{R}_k$ is invertible, from the first equality,

$$\mathrm{s}_k = \mathrm{R}_k^{-1} \cdot (-1_{K)} + t_{k+1}^{(k+1)} \mathrm{R}_k^{-1} \cdot (-\mathrm{g}_k)$$

$$(14) \quad \mathrm{s}_k = (\mathrm{t}^{(k)} + t_{k+1}^{(k+1)} \cdot \ell^{(k)}.$$

To determine the value of $t_{k+1}^{(k+1)}$ we substitute S_k in the second equation of (13) and get

$$(15) \quad t_{k+1}^{(k+1)} = -\frac{1 + \mathrm{u}_k^t \cdot \mathrm{t}^{(k)}}{\mathrm{u}_k^t \cdot \ell^{(k)} + \alpha_{k+1}^{k+1}}.$$

This, the solution of (9), given by (13), is

$$(16) \quad \mathrm{t}^{(k+1)} = \begin{pmatrix} \mathrm{t}^{(k)} \\ 0 \end{pmatrix} + t_{k+1}^{(k+1)} \begin{pmatrix} \ell^{(k)} \\ 1 \end{pmatrix}.$$

Lemma 2.2 $\mathrm{u}_k^t \cdot \ell^{(k)} + \alpha_{k+1}^{k+1} \neq 0.$

Proof: Since $\ell^{(k)}$ satisfies

$$\vartheta_k + \mathrm{V}_k \ell^{(k)} = 0$$

we build up a polynomial of degree k with k simple roots. In other words,

$$\alpha_k^k + \ell_1^k + \alpha_k \ell_2^k + \dots + \alpha_k^k \ell_k^k = 0$$

then

$$\alpha_{k+1}^{k+1} + \ell_1^k + \alpha_{k+1} \ell_2^k + \dots + \alpha_k^{k+1} \ell_k^k \neq 0$$

because $\alpha_i \neq \alpha_j, \quad \forall_i \neq j.$

From (10) and (15)

$$(17) \quad \ell_1^{(k+1)} \cdot t_{k+1}^{(k+1)} = 1.$$

Recalling (14), we know that

$$\mathrm{s}_k = (\mathrm{t}^{(k)} + t_{k+1}^{(k+1)} \cdot \ell^{(k)};$$

Multiplying by $\ell_1^{(k+1)}$, we obtain

$$(18) \quad \ell_1^{(k+1)} \cdot \mathrm{s}_k = \mathrm{h}_k$$

and from here

$$\ell_1^{(k+1)} \begin{pmatrix} \mathrm{s}_k \\ t_{k+1}^{(k+1)} \end{pmatrix} = \ell_1^{(k+1)} \mathrm{t}^{(k+1)} = \begin{pmatrix} \mathrm{h}_k \\ 1 \end{pmatrix}.$$

Therefore,

$$(19) \quad \mathrm{t}^{(k+1)} = \frac{1}{\ell_1^{(k+1)}} \cdot \begin{pmatrix} \mathrm{h}_k \\ 1 \end{pmatrix}$$

and

$$(20) \quad \begin{pmatrix} 1 \\ \mathrm{t}^{(k+1)} \end{pmatrix} = \frac{1}{\ell_1^{(k+1)}} \cdot \begin{pmatrix} \ell^{(k+1)} \\ 1 \end{pmatrix}$$

Lemma 2.3 $\ell_1^{(k+1)} \neq 0.$

Proof: If $\ell_1^{(k+1)} = 0,$ then the components $\ell_2^{(k+1)}, \ell_3^{(k+1)}, \cdots, \ell_{k+1}^{(k+1)}$, satisfy

$$\mathrm{V}(\ell_2^{(k+1)}, \ell_3^{(k+1)}, \cdots, \ell_{k+1}^{(k+1)})^T = -(\alpha_1^k, \alpha_2^k, \cdots, \alpha_{k+1}^k)^T$$

and

$$\mathrm{u}_k^t (\ell_2^{(k+1)}, \ell_3^{(k+1)}, \cdots, \ell_{k+1}^{(k+1)})^T = -\alpha_{k+1}^k$$

which is equivalent to say that the polynomial of degree k

$$p(x) = x^k + \ell_{k+1}^{(k+1)} x^{k-1} + \cdots + \ell_2^{(k+1)}$$

has α_i, for $i = 1, \cdots, k+1$ as simple roots.

Therefore, we rewrite (11) as

$$\ell^{(k+1)} = \begin{pmatrix} 0 \\ \ell^{(k)} \end{pmatrix} + \ell_1^{(k+1)} \cdot \begin{pmatrix} 1 \\ \mathrm{t}^{(k)} \end{pmatrix}$$

$$(21) \quad = \begin{pmatrix} 0 \\ \ell^{(k)} \end{pmatrix} + \ell_1^{(k+1)} \cdot \frac{1}{\ell_1^{(k)}} \begin{pmatrix} \ell^{(k)} \\ 1 \end{pmatrix}$$

Plug (20) into (10) and get

(22) $\ell_1^{(k+1)} = -\alpha_{k+1} \cdot \ell_1^{(k)}$.

In the same way, plugging (22) into (21) gives

(23) $$\ell^{(k+1)} = \begin{pmatrix} 0 \\ \ell^{(k)} \end{pmatrix} + (-\alpha_{k+1}) \cdot \begin{pmatrix} \ell^{(k)} \\ 1 \end{pmatrix}.$$

SOLVING THE GENERAL RIGHT HAND SIDE SYSTEM

We solve (3) for arbitrary **f** using recurrent relations between solutions of type (4) subsystems and solutions of higher order subsystems.

Let us suppose that the solution of the system

(24) $$V_k^t \cdot a^{(k)} = f^{(k)}$$

of order k is know, and

(25) $$V_k^t \cdot \ell^{(k)} = -\vartheta^{(k)};$$

we want to solve the system of order k+1

(26) $$V_{k+1}^t \cdot a^{(k+1)} = f^{(k+1)},$$

that is

$$\begin{pmatrix} V_k^t & \vartheta^{(k)} \\ r^{(k)} & \alpha_{k+1}^k \end{pmatrix} \begin{pmatrix} w \\ \alpha_{k+1}^{(k+1)} \end{pmatrix} = \begin{pmatrix} f^{(k)} \\ f_{k+1}^{(k+1)} \end{pmatrix}$$

where $\mathbf{r}^{(k)} = (1\alpha_{k+1} \cdots \alpha_{k+1}^{k-1})$ is a row vector and **w** represents the first k^{th} coordinates of $\mathbf{a}^{(k+1)}$.

By using block multiplications, (24), (25), and reasoning as previous section we obtain

(27) $$\begin{cases} V_k^t w + \alpha_{k+1}^{(k+1)} \vartheta^{(k)} = f^{(k)} \\ r^{(k)} w + \alpha_{k+1}^k \cdot \alpha_{k+1}^{(k+1)} = f_{k+1}^{(k+1)}. \end{cases}$$

Solving **w** from the first equation in (27) and using the fact that $a^{(k)} = V_k^{-t} f^{(k)}$ and $\ell^{(k)} = V_k^{-t}(-\vartheta^{(k)})$;; we have

(28) $$w = a^{(k)} + \alpha_{k+1}^{(k+1)} \ell^{(k)}.$$

In order to know $\alpha_{k+1}^{(k+1)}$, we plug w in last of equation (27); so by solving we obtain

(29) $$\alpha_{k+1}^{(k+1)} = \frac{f_{k+1}^{(k+1)} - r^{(k)} a^{(k)}}{r^{(k)} \ell^{(k)} + \alpha_{k+1}^k}.$$

Hence, the recurrent formula that solves (3) is

(30) $$a^{(k+1)} = \begin{pmatrix} a^{(k)} \\ 0 \end{pmatrix} = \alpha_{k+1}^{(k+1)} \begin{pmatrix} \ell^{(k)} \\ 1 \end{pmatrix}.$$

In conclusion, the solution of (3) for arbitrary **f** is reached from the solutions of

(31) $$\begin{cases} V_k^t a^{(k)} = f^{(k)} \\ V_k^t \ell^{(k)} = -\vartheta^{(k)}. \end{cases} \quad k = 1 \cdots, n$$

Algorithm. Given the vectors $\alpha=(\alpha_1 \ldots \alpha_n)$, $\mathbf{f}=(f_1, \ldots, f_n) \in \mathbb{R}^n$, the forecoming algorithm compute $\mathbf{a}=(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $\mathbf{V}^t \mathbf{a} = \mathbf{f}$, for any arbitrary vector **f**. We will use the notation $\alpha(i) = \alpha_i$, $f(i) = f_i$, and $\alpha(i) = \alpha_i$, $i = 1, \ldots, n$.

```
ℓ(1) = -α(1);
α(1) = f(1);
For j = 2 : n

% Finding the denominator of (29)
ef = ℓ(j-1) + α(j);
For i = 3 : j
ef = ef * α(j) + ℓ(j-i+1);
endfor;

% Computation of solution a
δ = 0;
For i = 1 : j-1
δ = α(j-i) + α(j)*δ;
endfor;

β = (f(j) - δ) /ef ;
α(1: j - 1) = α(1: j - 1) + β * ℓ(1: j - 1);
α(j) = β;

% Computation of particular solution
ℓ(j) = 1;
ℓ(j : - 1 : 2) = ℓ(j - 1 : - 1 : 1) - α(j) * ℓ(j: - 1 : 2);
ℓ(1) = - α(j) * ℓ(1);
endfor.
```

The foregoing algorithm needs $4n^2 - 2n - 2$ operations and $4n$ in memory space, which is considerably les that the used in Björck and Pereyra.

BIBLIOGRAPHYS REFERENCES

AZIZ A., WADIE, 1994. Resolución numérica de los Sistemas de Vandermonde, Tesis de Grado, Universidad de Los Andes, Facultad de Ciencias, Mérida-Venezuela, Julio.

BJÖRCK, A.; ELFVING T., 1973. Algorithm for Confluent Vandermonde Systems, Numerishe Mathematik 21, 130-137.

BJÖRCK, A.; PEREYRA V., 1970. Solution of Vandermonde Systems of Equations, Mathematics of Computation 12 No. 112, 893-903.

CHECA M., E.; MARQUÉS M., 2001. Álgebra Lineal Numérica: Teoría y Prácticas con Matemática, comunicación mathematica y c (i), Editorial Universidad Politécnica de Valencia, España.

GENE H., GOLUB; VAN LOAN, CHARLES F., 1996. Matrix Computations, third edition, The Johns Hopkins University Press, USA.

PORSANI, M., 1992. Levinson Type Algorithms for Hankel and Vandermonde Systems, II Escuela Latinoamericana de Geofisica.

SAAD, Y., 2003. Iterative Methods for Superlinear Systems, SIAM.

SHOKROLLHI, M.A.; OLSHEVSKY, V., 2001. A displacement Approach to Decoding Algebraic Codes in Fast Algorithms for Structural Matrices and Polynomials. Unified Superfast Algorithms, Birkhäuser.

TANG, W.P.,; GOLUB, G.H., 1981. The block Decomposition of a Vandermonde Matrix and its Applications, BIT 21, 505-517.